SEMI-POINTED PARTITIONS POSETS AND SPECIES

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ABSTRACT. We define semi-pointed partition posets, which are a generalisation of partition posets and show that they are Cohen-Macaulay. We then use multichains to compute the dimension and the character for the action of the symmetric groups on their homology. We finally study the associated incidence Hopf algebra, which is similar to the Faà di Bruno Hopf algebra.

Keywords: Poset, Incidence Hopf algebra, Möbius number, Partitions.

INTRODUCTION

The partition poset on a finite set V is the well-known partially ordered set, or *poset*, of partitions of V, endowed with the following partial order: a partition P is smaller than another partition Q if the parts of P are unions of parts of Q. A variant of partition posets, called *pointed partition posets*, has been studied by F. Chapoton and B. Vallette in [CV06] and [Val07]. A pointed partition of a set V is a partition of V, with a distinguished element for each of its parts. The pointed partition poset on V is then the set of pointed partitions of V, where a pointed partition P is smaller than another pointed partition Q if and only if the parts of P are unions of parts of Q and the set of pointed elements of P is included in the set of pointed elements of Q.

A variant of partition posets and pointed partition posets naturally appears during the study of intervals in some hypertree posets ([DO14]): we call them *semi-pointed partition posets*. The link between the homology of posets of partitions and pointed partitions and operads Lie and PreLie raises naturally the question of the study of the homology of semi-pointed partition posets. Indeed, B. Fresse proved in [Fre04] that the homology of the partition posets was isomorphic to the Koszul dual of the operad Comm, which is the operad Lie, tensorised by the signature representation. Moreover, F. Chapoton and B. Vallette proved in [CV06] that the homology of the pointed partition posets was isomorphic to the Koszul dual of the operad

Perm, which is the operad PreLie, tensorised by the signature representation. More generally, B. Vallette proved in [Val07] that the homology of a poset of partitions decorated by an operad is concentrated in higher degree if and only if the operad is Koszul and that, in this case, the homology of the poset is the Koszul dual of the operad, tensorised by the signature representation. The semi-pointed partition posets can then be seen as a poset of partition decorated by a coloured operad, which is a generalisation of classic operad, where the operad involved mixes both Comm and Perm. This case raises the question of the extension of B. Vallette's results to coloured operad. We intend to study this question later.

We recall in the first section the generalities on coloured operads, 2-species and poset homology. We also explain how the study of the homology of some posets can be reduced to the study of multichains. After a short description of semi-pointed partition posets, we then show in the second section that the posets are Cohen-Macaulay by proving their total semi-modularity. In the third section, we use the theory of species to compute the dimension of the unique non trivial homology group of a given semi-pointed partition poset. This dimension is given by the following theorem:

Theorem. *Let V_1 and V_2 be two finite sets of cardinality p and ℓ. The sum of dimensions of the unique homology group of every maximal interval in the semi-pointed partition poset over V_1 and V_2, whose minimums have a unique part which is pointed, is given by:*

$$\frac{(p+\ell-1)!}{(p-1)!}(p+\ell)^{p-1}.$$

The dimension of the unique homology group of the augmented poset of semi-pointed partitions over V_1 and V_2 is given by:

$$\frac{(p+\ell-1)!}{(p-1)!}(p+\ell-1)^{p-1}.$$

The sum of dimensions of the unique homology group of every maximal interval in the semi-pointed partition poset over V_1 and V_2 is given by:

$$\frac{(p+\ell-1)!}{p!}(p+\ell)^{p}.$$

The reasoning used in this section relies on the link between the homology of the poset and multichains in the poset. We then compute the action of the symmetric groups on the homology of the semi-pointed partition poset, which can be compared to the known characters of both Lie and PreLie operads.

We then compute the incidence Hopf algebra associated to the hereditary family generated by maximal intervals in semi-pointed partition posets. We prove that it is isomorphic to the Hopf algebra structure on functions on formal diffeomorphisms in dimension 2. It can therefore be seen as a generalisation of the Faà di Bruno Hopf algebra. This incidence Hopf algebra enables us to compute characteristic polynomials for maximal intervals

whose greatest element is pointed. Some of the results appearing here were announced in the extended abstract [DO15].

1   GENERALITIES

1.1   Species and operads

1.1.1   Species and 2-species. A *species* is a functor from the category of finite sets and bijections Bij to itself. A *2-species* is then a functor from Bij × Bij to Bij. We refer the reader to [BLL98] for more details about these objects.

Example 1.1. The map $\mathcal{A}_{r,s}$ which maps sets V_1 and V_2 with the set of forests of rooted trees whose roots are labelled by V_1 and whose other vertices are labelled by V_2 is a 2-species. It is represented on Figure 1.
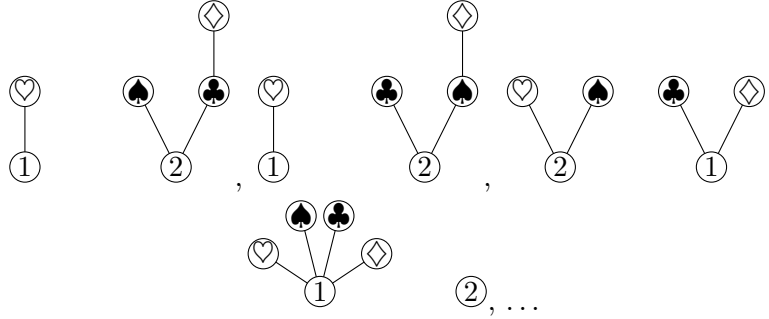

FIGURE 1. Elements of the image by the species $\mathcal{A}_{r,s}$ of the sets $\{1, 2\}$ and $\{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$

The map which associates to two sets V_1 and V_2 the set of sets (resp. lists, pointed sets) on $V_1 \sqcup V_2$ is a 2-species.

The following operations can be defined on 2-species.

Definition 1.1. Let $\mathcal{F}$ and $\mathcal{G}$ be two 2-species. The following operations are defined:

- $(\mathcal{F} + \mathcal{G})(I_1, I_2) = \mathcal{F}(I_1, I_2) \sqcup \mathcal{G}(I_1, I_2)$, (addition)
- $(\mathcal{F} \cdot \mathcal{G})(I, J) = \sum_{I_1 \sqcup I_2 = I, J_1 \sqcup J_2 = J} \mathcal{F}(I_1, J_1) \times \mathcal{G}(I_2, J_2)$, (product)
- If $\mathcal{E}$ is a species and $\mathcal{F}(\emptyset, \emptyset) = \emptyset$,

$$(\mathcal{E} \circ \mathcal{F})(I_1, I_2) = \bigsqcup_{\pi \in \mathcal{P}(I_1 \cup I_2)} \mathcal{E}(\pi) \times \prod_{J \in \pi} \mathcal{F}(J_1, J_2),$$

where $\mathcal{P}(I_1 \cup I_2)$ is the set of partitions of $I_1 \cup I_2$ and $J_i = J \cap I_i$. (substitution) We will sometimes omit $\circ$ and write $\mathcal{E}(\mathcal{F})$ for $\mathcal{E} \circ \mathcal{F}$.

Especially, $\mathbb{E} \circ \mathcal{F}$ will denote the substitution of $\mathcal{F}$ in the species of sets $\mathbb{E}$, defined by $\mathbb{E}(V) = V$ for any set V, which satisfies:

$$(\mathbb{E} \circ \mathcal{F})(I_1, I_2) = \bigsqcup_{\pi \in \mathcal{P}(I_1 \cup I_2)} \prod_{J \in \pi} \mathcal{F}(J_1, J_2).$$

To 2-species, the following generating series are also associated:

Definition 1.2. Let $\mathcal{F}$ be a 2-species. The (exponential) generating series associated with $\mathcal{F}$ is defined by:

$$\mathbf{C}_\mathcal{F}(x, y) = \sum_{n \geq 0} \sum_{m \geq 0} |\mathcal{F}(\{1, \ldots, n\}, \{1, \ldots, m\})| \frac{x^n y^m}{n! m!}.$$

The cycle index series associated with $\mathcal{F}$ is the formal series in an infinite number of variables $\mathbf{p} = (p_1, p_2, p_3, \ldots), \mathbf{q} = (q_1, q_2, q_3, \ldots)$ defined by:

$$\mathbf{Z}_\mathcal{F}(\mathbf{p}, \mathbf{q}) = \sum_{\substack{n, m \geq 0 \\ (n,m) \neq (0,0)}} \frac{1}{n! m!} \left(\sum_{\sigma \in \mathfrak{S}_n, \tau \in \mathfrak{S}_m} \mathcal{F}^{\sigma, \tau} p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \cdots q_1^{\tau_1} q_2^{\tau_2} q_3^{\tau_3} \cdots \right),$$

where $\mathcal{F}^{\sigma, \tau}$ is the number of $\mathcal{F}$-structures on $(\{1, \ldots, n\}, \{1, \ldots, m\})$ fixed by the action of σ on $\{1, \ldots, n\}$ and of τ on $\{1, \ldots, m\}$.

For $f = f(\mathbf{p})$ and $g = g(\mathbf{p}, \mathbf{q})$, to formal series in infinite variables $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ and $\mathbf{q} = (q_1, q_2, q_3, \ldots)$, the *plethystic substitution* $f \circ g$ is defined by:

$$kth \text{ term}$$
$$\downarrow$$
$$f \circ g(\mathbf{p}, \mathbf{q}) = f(\ldots, g(p_k, p_{2k}, p_{3k}, \ldots, q_k, q_{2k}, q_{3k}, \ldots), \ldots).$$

This substitution is linear and distributive on the left.

The suspension is defined as :

$$(1) \qquad \Sigma f(p_1, p_2, \ldots, q_1, q_2, \ldots) = -f(-p_1, -p_2, \ldots, -q_1, -q_2, \ldots).$$

These series satisfy the following relations:

Proposition 1.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two 2-species and $\mathcal{E}$ be a species. Their associated generating series satisfy:*
$$\mathbf{C}_{\mathcal{F}+\mathcal{G}} = \mathbf{C}_\mathcal{F} + \mathbf{C}_\mathcal{G}, \quad \mathbf{C}_{\mathcal{F} \cdot \mathcal{G}} = \mathbf{C}_\mathcal{F} \times \mathbf{C}_\mathcal{G},$$
$$\mathbf{Z}_{\mathcal{F}+\mathcal{G}} = \mathbf{Z}_\mathcal{F} + \mathbf{Z}_\mathcal{G}, \quad \mathbf{Z}_{\mathcal{F} \cdot \mathcal{G}} = \mathbf{Z}_\mathcal{F} \times \mathbf{Z}_\mathcal{G},$$
and if $\mathcal{G}(\emptyset, \emptyset) = \emptyset$, $\quad \mathbf{C}_{\mathcal{E} \circ \mathcal{G}} = \mathbf{C}_\mathcal{E} \circ \mathbf{C}_\mathcal{G}, \quad \mathbf{Z}_{\mathcal{E} \circ \mathcal{G}} = \mathbf{Z}_\mathcal{E} \circ \mathbf{Z}_\mathcal{G}.$

Example 1.3. The map $\mathcal{A}_{r,s}$ in Example 1.1 satisfies the following equation:

$$\mathcal{A}_{r,s} = \mathbb{E} \circ (X_1 \times \mathbb{E}(\mathcal{A}_2)),$$

where the 2-species X_1 and $\mathcal{A}_2$ are defined as follows: $X_1(V_1, V_2) = V_1$ if $|V_1| = 1$, $\emptyset$ otherwise and $\mathcal{A}_2(V_1, V_2) = \{\text{rooted trees on } V_2\}$.

1.1.2 Operads and coloured operads. We present in this paragraph the notion of operad and coloured operad. The reader can refer to [LV12] for more information about operads and to [VdL04] for more information about coloured operads. The name "operad" first appears in the article [May72] of P. May. This notion, introduced in the frame of iterated loop spaces, was then developed by other topologists such as M. Boardman and R. Vogt. In the context of homotopy theory, M. Boardman and R. Vogt especially introduced in the book [BV73] the notion of coloured operad, which generalizes the notion of operad.

A (symmetric) operad $\mathcal{O}$ is a species with an additive structure: a map from $\mathcal{O} \circ \mathcal{O}$ to $\mathcal{O}$ satisfying some additional axioms. We recall the equivalent usual definition before introducing the definition for coloured operads:

Definition 1.3. An *operad* is a sequence of sets $\mathcal{P}(n)$ ($n \in \mathbb{N}$), with a right action of the symmetric group $\mathfrak{S}_n$ on $\mathcal{P}(n)$, and with a distinguished element **1** in $\mathcal{P}(1)$ called the *identity*, together with composition maps

$$\circ : \mathcal{P}(n) \times \mathcal{P}(k_1) \times \ldots \times \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \ldots + k_n)$$
$$(\alpha, \alpha_1, \ldots, \alpha_n) \mapsto \alpha \circ (\alpha_1, \ldots, \alpha_n),$$

satisfying:

- associativity:

$$\alpha \circ (\alpha_1 \circ (\alpha_{1,1}, \ldots, \alpha_{1,k_1}), \ldots, \alpha_n \circ (\alpha_{n,1}, \ldots, \alpha_{n,k_n}))$$
$$= (\alpha \circ (\alpha_1, \ldots, \alpha_n)) \circ (\alpha_{1,1}, \ldots, \alpha_{1,k_1}, \ldots, \alpha_{n,1}, \ldots, \alpha_{n,k_n})$$

- identity:
$$\alpha \circ (\mathbf{1}, \ldots, \mathbf{1}) = \alpha = \mathbf{1} \circ \alpha$$

- equivariance: the composition commutes with the action of the symmetric group.

Example 1.4.
- Operad Comm is the operad which underlying species is the species of non empty sets $\mathbb{E} - 1$. The composition is given by considering a set of sets as a set.
- Operad Perm is the operad which underlying species is the species of pointed sets and which associates to a set an element of it (its pointed element). The composition is given by pointing the pointed element of the pointed set. For instance, $\{1, \mathbf{2}, 3\} \circ (\{\mathbf{1}\}, \{2, 3, \mathbf{4}\}, \{5, \mathbf{6}\})$ is $\{1, 2, 3, \mathbf{4}, 5, 6\}$.
- Operad PreLie is the operad which underlying species is the species of rooted trees, introduced in [CL01].
- Operad Lie is the Koszul dual of operad Comm. The suspension of its cycle index series is the inverse for the substitution of the cycle index series of Comm.

The notion of 2-coloured operads is a special case of the notion of coloured operads. It is given by the following definition:

Definition 1.4. A *2-coloured operad* is a sequence of sets $\mathcal{P}(c_1, \ldots, c_n; c)$ $(n \in \mathbb{N}; c_1, \ldots, c_n, c \in \{0,1\})$, with a right action of the symmetric group $\mathfrak{S}_n$, with a distinguished element $\mathbf{1}_0 \in \mathcal{P}(0;0)$ (resp. $\mathbf{1}_1 \in \mathcal{P}(1;1)$) called the *identity* on 0 (resp. 1), together with a composition map

$$\circ : \mathcal{P}(c_1, \ldots, c_n; c) \times \mathcal{P}(d_{1,1}, \ldots, d_{1,k_1}; c_1) \times \ldots$$
$$\times \, \mathcal{P}(d_{n,1}, \ldots, d_{n,k_n}; c_n) \rightarrow \mathcal{P}(d_{1,1}, \ldots, d_{n,k_n}; c)$$

satisfying associativity, identity and equivariance axioms. The diagrams involved are fully depicted in [EM06, §2].

The underlying sets, together with the action of the symmetric groups $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$, permuting elements of $V_1 \sqcup V_2$, but fixing V_1 and V_2, can be seen as a 2-species.

1.2 Poset homology The reader can refer to [Wac07] for an introduction on poset topology. We briefly recall here the notions. Let P be a finite poset.

Definition 1.5. A *strict m-chain* in the poset P is a m-tuple $(a_1, \ldots, a_m)$ where a_i are elements of P, neither maximum nor minimum in P, and $a_i \prec a_{i+1}$, for all $i \geq 1$. We write $\mathcal{C}_m(P)$ for the set of strict $(m+1)$-chains and $C_m(P)$ for the $\mathbb{C}$-vector space generated by all strict $(m+1)$-chains.

The set $\cup_{m \geq 0} \mathcal{C}_m(P)$ defines then a simplicial complex. The homology of P is defined as the homology of this simplicial complex.

Define the linear map $d_m : C_{m+1}(P) \rightarrow C_m(P)$ which maps a $(m+1)$-simplex to its boundary by :

$$d_m(a_0 < \ldots < a_m) = \sum_{i=0}^{m} (-1)^{i-1} a_0 < \ldots < a_{i-1} < a_{i+1} < \ldots < a_m.$$

These maps satisfy $d_{m-1} \circ d_m = 0$. The obtained pairs $(C_m(P), d_m)_{m>0}$ form a *chain complex*. Thus, we can define the homology of the poset.

Definition 1.6. The homology group of dimension m of the poset P is:

$$H_m(P) = \operatorname{Ker} d_m / \operatorname{Im} d_{m+1}.$$

We consider in this article the reduced homology, written $\tilde{H}_i$. This homology is obtained by adding a vector space $C_{-1}(P) = \mathbb{C}.e$, and the trivial linear map $d : C_0 \rightarrow C_{-1}$, which maps every singleton to the element e. The reduced homology differs from the homology only on the first homology group $\tilde{H}_0$. The first homology group then satisfies:

$$\dim(\tilde{H}_0(P)) = \dim(H_0(P)) - 1.$$

Dimensions of the homology groups satisfy the following well-known property:

Lemma 1.5 ([Wac07], Euler-Poincaré formula)**.** *The Euler characteristic χ (or equivalently Möbius numbers) of the homology satisfies:*

$$(2) \qquad \chi = \sum_{m \geq 0} (-1)^m \dim \tilde{H}_m(P) = \sum_{m \geq -1} (-1)^m \dim C_m(P).$$

This relation can be generalized to some characters. Let G be a finite group and $(P, \preceq)$ be a G-poset, i.e. a poset together with a G-action on its elements that preserves the partial order. The poset P is not necessarily bounded but will be required to be finite and pure, i.e. all its maximal strict chains have the same finite cardinality $\dim P$. Then the complex reduced homology groups $\tilde{H}_j(\mathbb{C})$ are G-modules (see [Wac07, 2.3] for details). We now explain how characters for the action of G on the homology groups can be obtained thanks to the character for the action of G on k-multichains. We will denote by $\chi_{\tilde{H}_i}$ the character for the action of G on the homology group $\tilde{H}_i$ of P, χ_i^s the character for the action of G on the vector space C_i spanned by strict $i + 1$-chains and by χ_k^l the character for the action of G on the vector space generated by k-multichains. This link relies on the Hopf trace formula [Wac07, Theorem 2.3.9]:

$$(3) \qquad \mu_\chi := \sum_{i=0}^{\dim P} (-1)^i \chi_{\tilde{H}_i} = \sum_{i=-1}^{\dim P} (-1)^i \chi_i^s$$

This formula is the heart of the reasoning below : the study of homology groups will be obtained through the study of chains. We introduce the notion of multichains:

Definition 1.7. A *k-multichain* in the poset P is a k-tuple $(a_1, \ldots, a_k)$ of elements of P such that $a_i \preceq a_{i+1}$ for all $1 \leq i \leq k - 1$.

The number of n-multichains is a polynomial in n called the *zeta polynomial*. This polynomial was introduced by R. Stanley in [Sta74] and further developed by P. Edelman in [Ede80]. The link between zeta polynomials and Möbius numbers can be found in [Sta12, Prop 3.12.1] for bounded posets and posets without any least or greatest element. We extend it here to posets with a least or a greatest element and write it in terms of group representations. This reasoning was already used in [Oge13], where it enabled the author to compute the action of the symmetric group on the unique homology group of the hypertree poset.

We now link multichains and strict chains.

Proposition 1.6. *The character for the action of G on the vector space generated by k-multichains of the pure finite poset P, $k \geq 1$, is given by:*

- *if P is bounded,*

$$(4) \qquad \chi_k^l = \sum_{i=-1}^{\dim P} \binom{k+1}{i+2} \chi_i^s,$$

- *if P is not bounded but has a least or a greatest element,*

$$(5) \qquad \chi_k^l = \sum_{i=-1}^{\dim P} \binom{k}{i+1} \chi_i^s,$$

- *otherwise,*

$$(6) \qquad \chi_k^l = \sum_{i=0}^{\dim P} \binom{k-1}{i} \chi_i^s.$$

Proof. The principle of this proof is to establish a one-to-one correspondence between k-multichains, and a pair (w, sc), where w is a word on the alphabet $\{0, 1\}$ and sc is a strict chain in the poset. We denote by $\hat{0}$ the least element and $\hat{1}$ the greatest one, if they exist.

Let us consider a k-multichain $(a_1, \ldots, a_k)$ in the poset P. When deleting repetitions and extrema in the chain, we obtain a strict chain $(a_{i_1}, \ldots, a_{i_s})$. Moreover, we encode the multichain with the word $(w_1, \ldots, w_k)$, where $w_i \in \{0, 1\}$ for all i and

- $w_1 = 0$ if $a_1 = \hat{0}$, 1 otherwise;
- $w_{j+1} = 0$ if $a_j = a_{j+1}$, 1 otherwise, if $1 \le j \le k-1$.

Case 1: If the poset is bounded, then the number of 1 in w corresponds to the number of elements in the strict chain sc, if the multichain does not contain the maximum, or is one more than the number of element in the strict chain sc otherwise. Moreover, if the multichain only contains extrema, sc is empty.

Case 2: If the poset contains exactly one extremum, then the number of 1 in w corresponds to the number of element in the strict chain sc. Moreover, if the multichain only contains k times the extremum, sc is empty.

Case 3: If the poset contains no extremum, then $w_1 = 1$. The number of 1 in w corresponds to the number of elements in the strict chain sc and the first letter of w is a 1. If sc is a strict i-chain, there are $\binom{k-1}{i}$ different possible words. Moreover, as $k \ge 1$, there is at least one element in sc.

As P is a G-poset, the partial order of P is preserved by the action of G. Then, if an element g of G sends the k-multichain $(a_1, \ldots, a_k)$ to the k-multichain $(b_1, \ldots, b_k)$, we have $b_i = g \cdot a_i$ for all i and g will send the associated strict chain $(a_{i_1}, \ldots, a_{i_s})$ to the chain $(b_{i_1}, \ldots, b_{i_s})$. As the elements a_{i_j} are pairwise different, so are the elements b_{i_j}. Moreover, as the action of G preserves the partial order, the extrema are fixed points for this action, so as none of the a_{i_j} are extrema, none of the b_{i_j} are extrema and the chain $(b_{i_1}, \ldots, b_{i_s})$ is strict. The bijection defined above is thus compatible with the G-action. $\qquad\square$

The right part of Expressions (4), (5) and (6) are polynomial in k: they are then well-defined for non-positive integers k. Using the value $\binom{-1}{i} = (-1)^i$, we obtain immediately:

Proposition 1.7. *The alternate sum of characters for the action of G on the homology of the poset P is given by:*

- *if P is bounded,*

$$\tag{7} \mu_\chi = \chi^l_{-2}$$

- *if P is not bounded but has a least or a greatest element,*

$$\tag{8} \mu_\chi = -\chi^l_{-1}$$

- *otherwise,*

$$\tag{9} \mu_\chi = \chi^l_0 - 1.$$

We apply this proposition in a particular case: when the poset is Cohen-Macaulay.

Definition 1.8. A finite pure poset P is *Cohen-Macaulay* if all its homology groups but the one in highest degree $\dim P$ vanish.

Example 1.8. The poset of subsets of a finite set I, ordered by inclusion, is Cohen-Macaulay. The partition posets and pointed partition posets are Cohen-Macaulay.

When the poset P is Cohen-Macaulay, the character for the action of G on its unique homology group is given by the alternating sum of characters for the action of G on vector spaces of strict chains, according to the Hopf trace formula [Wac07, Theorem 2.3.9]. We then obtain:

Proposition 1.9. *The character for the action of G on the unique homology group of the poset P is given by:*

- *if P is bounded,*

$$\tag{10} \chi_{\widetilde{H}_{\dim P}} = (-1)^{\dim P}\chi^l_{-2}$$

- *if P is not bounded but has a least or a greatest element,*

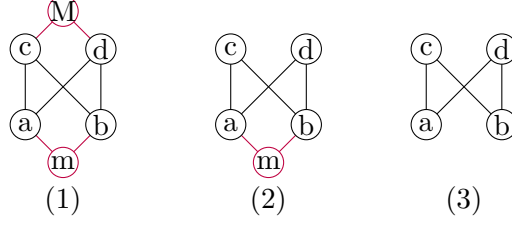$$\tag{11} \chi_{\widetilde{H}_{\dim P}} := (-1)^{\dim P - 1}\chi^l_{-1}$$

- *otherwise,*

$$\tag{12} \chi_{\widetilde{H}_{\dim P}} = (-1)^{\dim P}(\chi^l_0 - 1).$$

Example 1.10. Let us illustrate the formulae in the three following cases.

The Möbius number of these posets is $\mu = -1$. The number of strict chains satisfy $\chi^s_0 = 1$, $\chi^s_1 = 4$ and $\chi^s_0 = 2$. We then have:

(1) $\chi^l_k = 4 + 4(k - 1)$, and the equality $\mu = -1 + 4 + 4(-1)$ is satisfied
(2) $\chi^l_k = 1 + 4k + 2k(k - 1)$, and the equality $\mu = -(1 - 4 + 2 \times 2)$ is satisfied

(3) $\chi_k^l = k + 1 + 2k(k+1) + 4\frac{(k+1)k(k-1)}{6}$, and the equality $\mu = -2 + 1$
$+2(-2)(-1) + 4\frac{(-1)(-2)(-3)}{6}$ is satisfied

2   SEMI-POINTED PARTITIONS POSETS: PRESENTATION AND COHEN-MACAULAYNESS

Let us first define semi-pointed partition posets and show that their homology is concentrated in maximal degree.

2.1   Semi-pointed partitions posets Let V_1 and V_2 be two finite sets. The semi-pointed partition poset on V_1 and V_2 can be viewed as a poset of partitions on $V_1 \sqcup V_2$ decorated by a 2-coloured operad, which is a generalisation of the partition posets decorated by operads described in the article [Val07] of B. Vallette.

Let us first describe the involved 2-coloured operad:

Definition 2.1. The PSP operad is the 2-coloured operad defined by:
- $PSP(0,\ldots,0;1)$ and $PSP(1,\ldots,1;0)$ are empty, $PSP(c_1,\ldots,c_n;1)$ is the set of pointed sets $\{1,\ldots,\mathbf{i},\ldots,n\}$, pointed in a i satisfying $c_i = 1$, and $PSP(c_1,\ldots,c_n;0)$ is the set $\{\{1,\ldots,n\}\}$.
- The composition of a (possibly pointed) set E_1 in the element x of the (possibly pointed) set E_2 is:
    if E_2 is pointed in x: the set $E_1 \cup E_2 - \{x\}$ pointed in the pointed element of E_1 if there is one, or is not pointed otherwise,
    if E_2 is pointed in an element y and $x \neq y$: the set $E_1 \cup E_2 - \{x\}$ pointed in y
    otherwise: the non-pointed set $E_1 \cup E_2 - \{x\}$.

We keep the same notation PSP for the operad and the underlying species.

The semi-pointed partitions admit the two following equivalent definitions.

Definition 2.2. Let V_1 and V_2 be two sets of cardinality p and ℓ. A *semi-pointed partition* of (V_1, V_2) is a partition of $V_1 \sqcup V_2 = (P_1, \ldots, P_k)$ such that to each part P_j is associated an element of $PSP(V_1 \cap P_j, V_2 \cap P_j)$.

An alternative definition is that a *semi-pointed partition* is a partition of $V_1 \sqcup V_2$ such that each part in the partition satisfies:
- If all elements in the part belong to V_1, the part is pointed in one of its element,

- If all elements in the part belong to V_2, the part is not pointed,
- If some elements in the part belong to V_1 and other to V_2, the part can be not pointed or pointed in one of its elements belonging to V_1.

A (n, p)-semi-pointed partition is a semi-pointed partition of $V = \{1, \ldots, n\}$ with $V_1 = \{1, \ldots, p\}$ and $V_2 = \{p + 1, \ldots, n\}$.

We will write in bold the pointed element in each part.

Example 2.1. The set of $(3, 2)$-semi-pointed partitions is the following: $\{\mathbf{1}\}\{\mathbf{2}\}\{3\}$, $\{\mathbf{1}\}\{\mathbf{2}, 3\}$,$\{\mathbf{1}\}\{2, 3\}$, $\{\mathbf{2}\}\{\mathbf{1}, 3\}$, $\{\mathbf{2}\}\{1, 3\}$, $\{3\}\{\mathbf{1}, 2\}$, $\{3\}\{1, \mathbf{2}\}$, $\{\mathbf{1}, 2, 3\}$, $\{1, \mathbf{2}, 3\}$, $\{1, 2, 3\}$.

Let V be a finite set. The set of semi-pointed partitions on $V = V_1 \sqcup V_2$ can be endowed with the following partial order:

Definition 2.3. Let P and Q be two semi-pointed partitions. The partition P is smaller than the partition Q if and only if the parts of P are unions of parts of Q and the pointing of parts of P is "inherited" from the ones of Q, meaning that if a part p of P is union of parts $(q_1, \ldots, q_k)$ of Q, then the pointing of p is chosen in those of the q_i, given that if one of the q_i is not pointed, the part p can be not pointed. We will say that we *merge* the parts $(q_1, \ldots, q_n)$ into p.

Example 2.2. With $V_1 = \{1, 2, 3\}$ and $V_2 = \{4, 5\}$, the semi-pointed partition $\{\mathbf{1}, 2\}\{3, 4\}\{5\}$ is smaller than the semi-pointed partition $\{\mathbf{1}, 2, 3, 4\}\{5\}$ but cannot be compare to the semi-pointed partition $\{1, 2, \mathbf{3}, 4\}\{5\}$.

Remark 2.1. When the semi-pointed partitions are viewed as partitions decorated by the coloured operad PSP, the definition of the associated partial order can be seen as a generalisation of the order defined on partitions decorated by an operad to coloured operad: if P and Q are two semi-pointed partitions, with P smaller than Q, and if a part p of P is union of parts $(q_1, \ldots, q_n)$ of Q, the element of $PSP(p \cap V_1, p \cap V_2)$ chosen as a decoration of the part p is chosen in the composition of the decorations.

We denote by $\widehat{\Pi_{V,V_1}}$ the poset of semi-pointed partitions of $V = V_1 \sqcup V_2$ bounded by the addition of a least element $\widehat{0}$ and $\widehat{\Pi_{n,p}}$ the poset of (n, p)-semi-pointed partitions bounded by the addition of a smallest element $\widehat{0}$. The maximal intervals in $\Pi_{n,p}$ whose least element is pointed are all isomorphic: we write $\Pi_{n,p}^1$ the maximal interval in $\Pi_{n,p}$ whose least element is pointed in 1. We also write $\Pi_{n,p}^0$ for the maximal interval in $\Pi_{n,p}$ whose least element is not pointed. The partition whose parts are all of size 1, endowed with the unique way to point the parts, will be denoted by π_{V,V_1}. It is the greatest element of Π_{V,V_1}.

Let us remark that two semi-pointed partitions P and Q can be in several different posets Π_{V,V_1}. If the partition P is inferior to the partition Q in one of these posets, then it is the case in all other posets containing these two elements: the order between these elements does not depend on the underlying sets V_1 and V_2.

Remark 2.2. The poset $\Pi_{n,0}^0$ is the poset of partitions of $\{1, \ldots n\}$.

The posets $\Pi_{n,n}$ and $\Pi_{n,n-1}$ are two posets isomorphic to the pointed partition poset. This isomorphism comes by definitions for $\Pi_{n,n}$ and by identifying non pointed parts with parts pointed in the last element n for $\Pi_{n,n-1}$.
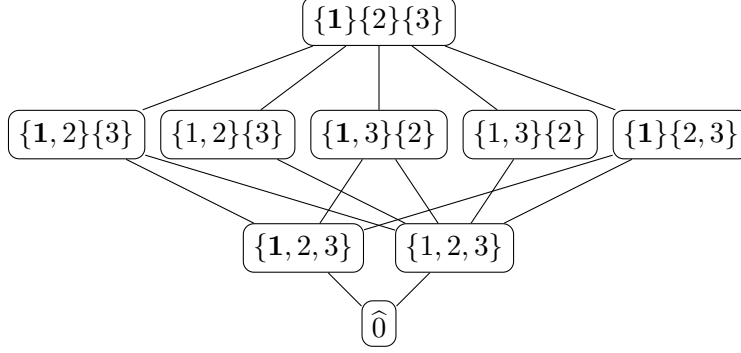

FIGURE 2. Poset $\widehat{\Pi_{3,1}}$.

2.2   Cohen-Macaulayness A poset P is *totally semi-modular* if for any interval I in P, and for any elements x, y in I which cover an element z in I, then there exists an element t in I covering x and y. It follows from [BW83] that any bounded, graded totally semi-modular poset is Cohen-Macaulay, i.e. has its homology concentrated in the highest degree. We then prove the following proposition by total semi-modularity:

Proposition 2.3. *The duals of the posets $\widehat{\Pi_{n,p}}$, $\Pi_{n,p}^0$ and $\Pi_{n,p}^1$ are totally semi-modular and thus the posets $\widehat{\Pi_{n,p}}$, $\Pi_{n,p}^0$ and $\Pi_{n,p}^1$ are Cohen-Macaulay (by [BW83, Corollary 5.2]).*

Proof. Let $[a, b]$ be an interval of the dual $\widehat{\Pi_{n,p}}^*$ of $\widehat{\Pi_{n,p}}$. The greatest element of $\widehat{\Pi_{n,p}}^*$ is denoted by $\widehat{1}$. Let us suppose that there exist three elements t, x, y in $[a, b]$ such that x and y cover t and such that x is different from y. We have to show the existence of an element z in $[a, b]$ covering x and y.

If x and y have only one part, $b = \widehat{1}$ and the elements x and y are covered by b, so we can choose $z = b$.

Otherwise, we can consider that x is obtained from t by merging the parts p_1 and p_2 of t with a pointed element e_x, if the part is pointed. The element e_x, if it exists, can be the pointed element of p_1, if it exists, or the pointed element of p_2, if it exists. We can also consider that y can be obtained by merging the parts p_3 and p_4 of t with a pointed element e_y, if the part is pointed. The element e_x, if it exists, can be the pointed element of p_3, if it exists, or the pointed element of p_4, if it exists. In the following, "choosing

the element e_x for a part p", will mean that either e_x exists and e_x is pointed in p or that p is not pointed.

- <u>First case</u>: The parts p_1, p_2, p_3 and p_4 are pairwise different.

  Then, we consider the element z obtained from t by merging p_1 and p_2 and choosing the pointed element e_x for the union of the parts, and by merging p_3 and p_4 and choosing the pointed element e_y for the union of the parts. This element z can be obtained from x by merging p_3 and p_4 and choosing the element e_y for the union of parts and from y by merging p_1 and p_2 and choosing the element e_x for the union of parts: z thus covers both x and y and belongs to the interval $[a, b]$.

- <u>Second case</u>: The parts p_1, p_2, p_3 and p_4 are not pairwise different.

  Then, up to a renumbering of parts, we can consider that p_1 and p_4 are equal. Note that there cannot be more than two equal parts as x and y are different and parts p_{2i-1} and p_{2i} are also different, for $i \in \{1, 2\}$. Let us now find an element covering both x and y. The underlying partition of such an element will be obtained from t by merging p_1, p_2 and p_3 in one part. We now have to choose a compatible pointed element for this new part in order to cover x and y and to be inferior to b.

  On the one hand: If $b \neq \widehat{1}$ and the pointed element of one part in b comes from a pointed element in a part p_i of p, as b is greater than x and y, the three parts p_1, p_2 and p_3 are in only one part of b, pointed in e_p. We then choose z to be the partition obtained from t by merging the parts p_1, p_2 and p_3 and by choosing the pointed element e_p. The partition z covers x and y and is smaller than b. Indeed, e_p can be chosen as the pointed element for the union of $p_1 \cup p_2$ and p_3 in x and for the union of p_2 and $p_1 \cup p_3$ in y. Moreover, the parts of b can be obtained as union of parts of z.

  On the other hand: Otherwise, we can choose the element z obtained from t by merging the three parts p_1, p_2 and p_3 and choosing their pointed element as follows:
    - If x and y both have inherited of the pointed element in p_1, z inherits of it.
    - Otherwise, if x or y has inherited of the pointed element in p_i, for $i \neq 1$, z inherits of the pointed element in p_i.

  $\square$

We study the unique non trivial homology group associated to semi-pointed partition posets in the following section.

3 HOMOLOGY OF THE SEMI-POINTED PARTITION POSET

We now apply the result 1.9 of section 1.2 to semi-pointed partition posets. In the unbounded poset of semi-pointed partitions on (V_1, V_2) Π_{V, V_1}, a strict

k-chain (resp. k-multichains) *with multiplicity* is a strict k-chain (resp. k-multichains) in one of the maximal interval of the poset. It is equivalent to the data of $(Ch_k, \min)$, where min is a partition in only one part of the poset and Ch_k is a strict chain (resp. multichain) of the poset whose elements are greater than min.

For $k \geq 1$, we denote by:

- $\mathcal{C}_k^g$ the species of k-multichains in semi-pointed partitions posets
- $\mathcal{C}_k^l$ the species of k-multichains with multiplicity in semi-pointed partitions posets

The generating series and cycle index series associated to these species are respectively given by $\mathbf{C}_k^g$, $\mathbf{C}_k^l$ and $\mathbf{Z}_k^g$ and $\mathbf{Z}_k^l$.

Hence, applying Proposition 1.9, we obtain:

Proposition 3.1. *The action of $\mathfrak{S}_p \times \mathfrak{S}_{n-p}$ induced on the unique homology group of the poset $\widehat{\Pi_{n,p}}$ is given by the opposite of the value in $k = -1$ of the polynomial $\mathbf{Z}_k^g$. The action of $\mathfrak{S}_p \times \mathfrak{S}_{n-p}$ induced on the direct sum of the homology of maximal intervals in the poset $\Pi_{n,p}$ is given by the value in $k = -2$ of the polynomial $\mathbf{Z}_k^l$.*

The aim of this section is then the computation of the dimension and the action of the symmetric groups on the unique non trivial homology group of $\widehat{\Pi_{n,p}}$.

3.1 Relations between species Let k be a positive integer.

We will need the following auxiliary species:

- $\mathcal{C}_k^\bullet$, the species which associates to (V_1, V_2) the set of k-multichains in $\Pi_{V_1 \cup V_2, V_1}$, whose minimum is a partition with only one part, which is pointed, (called k-pm-multichains)
- $\mathcal{C}_k^\times$, the species which associates to (V_1, V_2) the set of k-multichains in $\Pi_{V_1 \cup V_2, V_1}$, whose minimum is a partition with only one part, which is not pointed (called k-um-multichains).

The generating series and cycle index series associated to these species are $\mathbf{C}_k^\bullet$, $\mathbf{C}_k^\times$ and $\mathbf{Z}_k^\bullet$ and $\mathbf{Z}_k^\times$.

The species are linked by the following relations.

Proposition 3.2. *The species $\mathcal{C}_k^\bullet$, $\mathcal{C}_k^\times$, $\mathcal{C}_k^g$ and $\mathcal{C}_k^l$ are linked by:*

$$\mathcal{C}_k^\bullet = \mathcal{C}_{k-1}^\bullet \times \mathbb{E} \circ \left(\mathcal{C}_{k-1}^\bullet + \mathcal{C}_{k-1}^\times \right),$$

$$\mathcal{C}_k^\times = (\mathbb{E} - 1) \circ \mathcal{C}_{k-1}^\times \times \mathbb{E} \circ \mathcal{C}_{k-1}^\bullet,$$

$$\mathcal{C}_k^g = (\mathbb{E} - 1) \circ \left(\mathcal{C}_k^\bullet + \mathcal{C}_k^\times \right),$$

$$\mathcal{C}_{k-1}^l = \mathcal{C}_k^\bullet + \mathcal{C}_k^\times.$$

Proof.
- Consider a k-pm-multichain. Then, the element e following this minimum in the chain has a pointed part whose pointed element p is pointed in the minimum of the chain, and other parts pointed

or not. Forgetting the minimum in the chain and splitting parts of the partition e, we obtain a $(k-1)$-pm-multichain, whose minimum is pointed in p and a (eventually empty) set of $(k-1)$-multichains, whose minimum has only one part, pointed or not.

- Consider a k-um-multichain. Then, the element e following this minimum in the chain has at least a non-pointed part. Forgetting the minimum in the chain and splitting parts of the partition e, we obtain a non-empty set of $(k-1)$-um-multichains and a set of $(k-1)$-pm-multichains.

- Consider a k-multichain. The relation is obtained by splitting parts in the minimum of the chain.

- Consider a k-multichain with multiplicity $((a_1,\ldots,a_k),\mathfrak{m})$ of semi-pointed partitions in a maximal interval $[\mathfrak{m},\pi_{V,V_1}]$ of the unbounded poset. We obtain a $k+1$-multichain by adding at the beginning of the chain the partition $\mathfrak{m}$ in one part bounding the interval. The unique part of the partition $\mathfrak{m}$ can be bounded or not. There is then a one-to-one correspondence between the pairs $((a_1,\ldots,a_k),\mathfrak{m})$ and the chains $(\mathfrak{m},a_1,\ldots,a_k)$. The chain $(\mathfrak{m},a_1,\ldots,a_k)$ is a $(k+1)$-multichain whose minimum has only one part.

$\square$

3.2 Dimension of the homology of the semi-pointed partition poset We now want to compute the dimension of the unique homology group of semi-pointed partition posets and of their maximal intervals. We use the relations between species of Proposition 3.2: these relations imply relations between generating series which are written in the following proposition. These relations will give us closed explicit formulae for the dimension of the homologies. These relations are true for all positive integer k. However, the coefficient of series $\mathbf{C}_k^\bullet$, $\mathbf{C}_k^\times$, $\mathbf{C}_k^g$ and $\mathbf{C}_k^l$ are polynomial in k: the relations are then true for all integers.

Proposition 3.3. *For all integers k, the generating series $\mathbf{C}_k^\bullet$, $\mathbf{C}_k^\times$, $\mathbf{C}_k^g$ and $\mathbf{C}_k^l$ satisfy the following relations:*

$$(13) \qquad \mathbf{C}_k^\bullet = \mathbf{C}_{k-1}^\bullet \times e^{\mathbf{C}_{k-1}^\bullet + \mathbf{C}_{k-1}^\times},$$

$$(14) \qquad \mathbf{C}_k^\times = e^{\mathbf{C}_{k-1}^\bullet} \left(e^{\mathbf{C}_{k-1}^\times} - 1 \right),$$

$$(15) \qquad \mathbf{C}_k^g = \exp\left(\mathbf{C}_k^\bullet + \mathbf{C}_k^\times \right) - 1,$$

$$(16) \qquad \mathbf{C}_{k-1}^l = \mathbf{C}_k^\bullet + \mathbf{C}_k^\times.$$

Computing the first terms, we obtain:

Proposition 3.4. *The series $\mathbf{C}_{-1}^\bullet$ and $\mathbf{C}_{-1}^g$ are linked by:*

$$(17) \qquad \begin{cases} x = \mathbf{C}_{-1}^\bullet \left(1 + \mathbf{C}_{-1}^g \right), (a) \\ y = \mathbf{C}_{-1}^g + 1 - e^{\mathbf{C}_{-1}^\bullet}.(b) \end{cases}$$

$p \mid \ell$	0	1	2	3	4	5
0	0	1	2	5	15	52
1	1	3	8	25	89	354
2	3	10	35	133	552	2493
3	10	41	173	768	3637	
4	41	196	953	4815		
5	196	1057	5785			
6	1057	6322				
7	6322					

TABLE 1. Number of semi-pointed partitions in $\Pi_{n,p}$

Proof. We first compute $\mathbf{C}_1^\bullet$ and $\mathbf{C}_1^\times$. Let us recall that the coefficient of $\frac{x^p y^\ell}{p!\ell!}$ in $\mathbf{C}_1^\bullet$ (resp. $\mathbf{C}_1^\times$) corresponds to the number of 1-pm-multichains (resp. 1-um-multichains), on a set of "pointable" elements of size p and a set of "non-pointable" elements of size ℓ. There are p such elements in the pointed case of $\mathbf{C}_1^\bullet$ and 1 if ℓ is positive, 0 otherwise, in the non-pointed case $\mathbf{C}_1^\times$. We thus obtain:

$$\mathbf{C}_1^\bullet = xe^{x+y} \text{ and } \mathbf{C}_1^\times = e^{x+y} - e^x.$$

Using relations between generating series (13) and (14), we obtain:

$$(18) \qquad \begin{cases} xe^{x+y} = \mathbf{C}_0^\bullet e^{\mathbf{C}_0^\bullet + \mathbf{C}_0^\times} \\ e^{x+y} - e^x = e^{\mathbf{C}_0^\bullet + \mathbf{C}_0^\times} - e^{\mathbf{C}_0^\bullet}. \end{cases}$$

These equations imply the following functional equation which uniquely determines $\mathbf{C}_0^\bullet$:

$$xe^{x+y} = \mathbf{C}_0^\bullet \left(e^{x+y} - e^x + e^{\mathbf{C}_0^\bullet} \right).$$

Solving this equation and reporting it in 18, we obtain:

$$\mathbf{C}_0^\bullet = x \text{ and } \mathbf{C}_0^\times = y.$$

These values together with (13) and (14) for $k = 0$, with $\mathbf{C}_{-1}^\times$ eliminated thanks to (15), give the relations of the proposition.

$\square$

Remark 3.1. The coefficient of $\frac{x^p y^\ell}{p!\ell!}$ in the series $\mathbf{C}_1^g$ gives the number of semi-pointed partitions in $\Pi_{n,p}$. This series satisfies:

$$\mathbf{C}_1^g = \exp\left((x+1)e^{x+y} - e^x \right) - 1.$$

We summarize in Table 1 the first values for this series.

The first line corresponds to the case of partitions of a set of size n: this cardinality is given by the nth Bell number B_n. The first column $\ell = 0$ corresponds to the case of pointed partitions which are enumerated by $\sum_{p=0}^n \binom{n}{p}(n-p)^p$.

Equations (17) give the following equation for $\mathbf{C}^\bullet_{-1}$:

$$(19) \qquad x = \mathbf{C}^\bullet_{-1}\left(y + e^{\mathbf{C}^\bullet_{-1}}\right).$$

We now use Lagrange's inversion theorem, which can be found in [Sta01, Theorem 5.4.2, Corollary 5.4.3] for instance, to obtain the result.

Theorem 3.5. *Let V_1 be a set of cardinality p and V_2 be a set of cardinality ℓ. The sum of dimensions of the unique homology group of every maximal interval, whose minimum has a unique part which is pointed, in the semi-pointed partition poset over V_1 and V_2 is given by:*

$$\frac{(p+\ell-1)!}{(p-1)!}(p+\ell)^{p-1}.$$

The dimension of the unique homology group of the augmented poset of semi-pointed partitions over V_1 and V_2 is given by:

$$(-1)^{p+\ell-1}\frac{(p+\ell-1)!}{(p-1)!}(p+\ell-1)^{p-1}.$$

The sum of dimensions of the unique homology group of every maximal interval in the semi-pointed partition poset over V_1 and V_2 is given by:

$$(-1)^{p+\ell-1}\frac{(p+\ell-1)!}{p!}(p+\ell)^{p}.$$

Proof.
- Let us now compute the coefficients of $\mathbf{C}^\bullet_{-1}$. We apply to (19) Lagrange's inversion theorem with $F(x) = x\,(y + e^x)$. We get:

$$\mathbf{C}^\bullet_{-1} = \sum_{p\geq 1}\frac{x^p}{p!}\left(\frac{\partial}{\partial z}\right)^{p-1}\left(\frac{1}{y+e^z}\right)^p_{z=0},$$

where $f(z)_{z=0}$ stands for the value in $z=0$ of the map $f(z)$. The computation of the derivative, using Binomial theorem, gives:

$$\left(\frac{\partial}{\partial z}\right)^{p-1}\left(\frac{1}{y+e^z}\right)^p = \sum_{\ell\geq 0}\binom{-p}{\ell}y^\ell\left(\frac{\partial}{\partial z}\right)^{p-1}e^{z(-p-\ell)}.$$

We hence obtain the following equation:

$$\left(\frac{\partial}{\partial z}\right)^{p-1}\left(\frac{1}{y+e^z}\right)^p_{z=0} = \sum_{\ell\geq 0}(-1)^{p+\ell-1}\frac{(p+\ell-1)!}{(p-1)!}\frac{y^\ell}{\ell!}(p+\ell)^{p-1}.$$

The result immediately follows from this equation.
- The second equation follows from Lagrange's inversion theorem with $H(x) = y - 1 + e^x$ using (17)(b). The coefficient $\frac{x^p y^l}{p!l!}$ of the series we want to compute is then given by the coefficient of $\frac{x^{p-1}y^l}{(p-1)!l!}$ in:

$$H'(x)\left(\frac{x}{F(x)}\right)^p = e^x(y+e^x)^{-p}.$$

- The third equation is obtained thanks to the differential equation of the following lemma:

Lemma 3.6. *The generating series* $\mathbf{C}^l_{-2}$ *and* $\mathbf{C}^\bullet_{-1}$ *satisfy the following differential equation:*

$$x\frac{\partial \mathbf{C}^l_{-2}}{\partial x} = x\frac{\partial \mathbf{C}^\bullet_{-1}}{\partial x} + y\frac{\partial \mathbf{C}^\bullet_{-1}}{\partial y}.$$

Writing $\mathbf{C}^l_{-2} = \sum_{p\geq 0, \ell \geq 0, (p,\ell)\neq(0,0)} a_{p,\ell}\frac{x^p y^\ell}{p!\ell!}$, the above differential equation of Lemma 3.6 implies the following relation, for all $p \geq 0, \ell \geq 0, (p,\ell) \neq (0,0)$:

$$pa_{p,\ell} = -\frac{(p+\ell-1)!}{(p-1)!}(p+\ell)^{p-1}(p+\ell).$$

The result immediately follows.

$\square$

We now prove Lemma 3.6 used in the proof above.

Proof of Lemma 3.6. Using (16), we get:

$$x\frac{\partial \mathbf{C}^l_{-2}}{\partial x} = x\frac{\partial \mathbf{C}^\bullet_{-1}}{\partial x} + x\frac{\partial \mathbf{C}^\times_{-1}}{\partial x}.$$

The lemma is then true if and only if the following equality holds:

$$y\frac{\partial \mathbf{C}^\bullet_{-1}}{\partial y} = x\frac{\partial \mathbf{C}^\times_{-1}}{\partial x}.$$

To obtain this equation, we differentiate Equation (19) with respect to x and y. We obtain:

$$y\frac{\partial \mathbf{C}^\bullet_{-1}}{\partial y} = \frac{-y\mathbf{C}^\bullet_{-1}}{y + e^{\mathbf{C}^\bullet_{-1}} + \mathbf{C}^\bullet_{-1}e^{\mathbf{C}^\bullet_{-1}}}$$

and

$$\frac{\partial \mathbf{C}^\bullet_{-1}}{\partial x} = \frac{1}{y + e^{\mathbf{C}^\bullet_{-1}} + \mathbf{C}^\bullet_{-1}e^{\mathbf{C}^\bullet_{-1}}}.$$

We also differentiate Equation (17)(b) with respect to x. We get:

$$\frac{\partial \mathbf{C}^\times_{-1}}{\partial x} = \frac{\frac{\partial \mathbf{C}^\bullet_{-1}}{\partial x}\left(e^{\mathbf{C}^\bullet_{-1}} - e^{\mathbf{C}^\bullet_{-1}+\mathbf{C}^\times_{-1}}\right)}{e^{\mathbf{C}^\bullet_{-1}+\mathbf{C}^\times_{-1}}}.$$

However, according to Equation (17)(b), we have $e^{\mathbf{C}^\bullet_{-1}} - e^{\mathbf{C}^\bullet_{-1}+\mathbf{C}^\times_{-1}} = -y$ and according to Equation (17)(a), we have $e^{-\mathbf{C}^\bullet_{-1}-\mathbf{C}^\times_{-1}} = \frac{\mathbf{C}^\bullet_{-1}}{x}$, hence the result.

$\square$

3.3   Action of the symmetric groups on the homology of the semi-pointed partition poset The permutation of the elements of $V_1 \cup V_2$ induced by an element of $\mathfrak{S}_{V_1} \times \mathfrak{S}_{V_2}$ implies a permutation of semi-pointed partitions with the same number of parts. This permutation preserves the order and then induces an action on the homology of the semi-pointed partition poset on V_1 and V_2. The equations on species of Proposition 3.2 enable us to compute this action.

Theorem 3.7. *Let σ be a permutation of $\mathfrak{S}_{V_1} \times \mathfrak{S}_{V_2}$ with λ_i i-cycles on V_1 and μ_j j-cycles on V_2. The character for the action of the symmetric groups on the sum of homologies of the maximal intervals whose least elements are pointed is given on σ by:*

$$(20) \quad \left((-1)^{\lambda_1 + \mu_1 - 1} \frac{(\lambda_1 + \mu_1 - 1)!}{(\lambda_1 - 1)!} (\lambda_1 + \mu_1)^{\lambda_1 - 1} \right)$$

$$\times \prod_{l \geq 2} (-1)^{\mu_l + \lambda_l - 1} \frac{(\lambda_l + \mu_l - 1)!}{\lambda_l!} l^{\lambda_l} \alpha_l^{\lambda_l - 1} \left(-\alpha_l \lambda_l + (\lambda_l + \mu_l) l \lambda_l \right),$$

where $\alpha_m = \sum_{n \mid m} n(\lambda_n + \mu_n)$.

The character which evaluated in $t = 0$ gives the action of the symmetric groups on the sum of homologies of the maximal intervals, and which evaluated in $t = 1$ gives the action of the symmetric groups on the augmented semi-pointed partition posets, is given on σ by:

$$(21)$$
$$\frac{1}{t} \prod_m (-1)^{\lambda_m + \mu_m - 1} m^{\mu_m} \mu_m! \, (\alpha_m - g_{m,t})^{\lambda_m - 1} \left[m \lambda_m \binom{\lambda_m + \mu_m - g_{m,t}}{\mu_m} \right.$$
$$\left. - (\alpha_m - g_{m,t}) \binom{\lambda_m + \mu_m - g_{m,t} - 1}{\mu_m} \right]$$

where the sum runs over the integers m such that $\lambda_m + \mu_m > 0$, $\alpha_m = \sum_{n \mid m} n(\lambda_n + \mu_n)$, μ is the usual number theory Möbius function and $g_{m,t} = \frac{1}{m} \sum_{k \mid m} \mu(k) t^{m/k}$.

The evaluation in $t = 0$ of the preceeding character, which corresponds to the action of the symmetric groups on the sum of homologies of the maximal intervals, is given on σ by:

$$(22) \quad \left((-1)^{\lambda_n + \mu_n - 1} n^{\mu_n} \alpha_n^{\lambda_n - 1} \frac{\mu(n)(\lambda_n + \mu_n - 1)!}{n \times \lambda_n!} \left[\lambda_n(1 - n) + \alpha_n \right] \right) \times$$

$$\prod_{\substack{m > n \\ \lambda_m + \mu_m > 0}} (-1)^{\lambda_m + \mu_m - 1} m^{\mu_m} \alpha_m^{\lambda_m - 1} \frac{(\lambda_m + \mu_m - 1)!}{\lambda_m!} \lambda_m \left[m(\lambda_m + \mu_m) - \alpha_m \right]$$

where $\alpha_m = \sum_{n \mid m} n(\lambda_n + \mu_n)$, μ is the usual number theory Möbius function and $n = \min\{m \mid \alpha_m = m(\lambda_m + \mu_m)\}$.

Proof. (1) According to the equations of Proposition 3.2, the cycle index series of the species $\mathcal{C}^{\bullet}_{-1}$ satisfies:

$$p_1 = \mathbf{Z}^{\bullet}_{-1}(q_1 + \mathbb{E} \circ \mathbf{Z}^{\bullet}_{-1}),$$

where the cycle index series of $\mathbb{E}$ is $\exp(\sum_{k \geq 1} \frac{p_k}{k})$.

We want to compute the coefficient of $\prod_{i,j} p_i^{\lambda_i} q_j^{\mu_j}$ in $\mathbf{Z}^{\bullet}_{-1}$, hence the residue:

$$I = \int \mathbf{Z}^{\bullet}_{-1} \prod_{i,j \geq 1} \frac{dp_i}{p_i^{\lambda_i + 1}} \frac{dq_j}{q_j^{\mu_j + 1}}.$$

We use the change of variables $z_l = p_l \circ \mathbf{Z}^{\bullet}_{-1}$. Then z_l satisfy:

$$p_l = z_l \left(q_l + \exp \left(\sum_{k \geq 1} \frac{z_{kl}}{k} \right) \right)$$

and we obtain:

$$I = \int z_1 \prod_{l \geq 1, j} \left(\int \frac{q_l + \exp \left(\sum_{k \geq 1} \frac{z_{kl}}{k} \right) + z_l \exp \left(\sum_{k \geq 1} \frac{z_{kl}}{k} \right)}{\left(q_l + \exp \left(\sum_{k \geq 1} \frac{z_{kl}}{k} \right) \right)^{\lambda_l + 1}} \frac{dq_l}{q_l^{\mu_l + 1}} \right) \frac{dz_j}{z_j^{\lambda_j + 1}}$$

We compute

$$\int \frac{q_l + \exp \left(\sum_{k \geq 1} \frac{z_{kl}}{k} \right) + z_l \exp \left(\sum_{k \geq 1} \frac{z_{kl}}{k} \right)}{\left(q_l + \exp \left(\sum_{k \geq 1} \frac{z_{kl}}{k} \right) \right)^{\lambda_l + 1}} \frac{dq_l}{q_l^{\mu_l + 1}}$$

$$= \exp \left(\sum_{k \geq 1} \frac{z_{kl}}{k} (-\lambda_l - \mu_l) \right),$$

Hence, the result (20).

(2) To obtain this result, we have to compute the coefficient of $\prod_{i,j} p_i^{\lambda_i} q_j^{\mu_j}$ in

$$\frac{\mathbb{E} \circ t\mathbf{Z}^l_{-2} - 1}{t}.$$

We recall the notations of Example 1.4 : the cycle index series of operad Comm is equal to the cycle index series $\mathbb{E} - 1$ and its inverse for the substitution is operad Lie. Using relations between species established in Proposition 3.2, we obtain the following equality, stated in term of cycle index series.

$$\frac{\text{Comm} \circ t\mathbf{Z}^l_{-2}}{t} = \text{Comm} \circ t\Sigma \, \text{Lie} \circ \left(q_1 + \text{Comm} \circ \mathbf{Z}^{\bullet}_{-1} \right).$$

The substitution $\text{Comm} \circ t\Sigma \, \text{Lie}$ is computed in [Get95]. We then have to compute the following residue

$$\frac{1}{t} \int \prod_l \left(q_l + \exp \left(\sum_k \frac{p_{kl}}{k} \circ \mathbf{Z}^{\bullet}_{-1} \right) \right)^{g_{l,t}} \prod_{i,j} \frac{dp_i dq_j}{p_i^{\lambda_i + 1} q_j^{\mu_j + 1}},$$

where $g_{l,t} = \frac{1}{l} \sum_{k|l} \mu(k) t^{l/k}$, with μ the usual Möbius function of an integer.

We use the change of variable $z_l = p_l \circ \mathbf{Z}^{\bullet}_{-1}$ of the precedent point. We then use Newton binomial series and integrate with respect to the variables q_l.

Then, denoting $\alpha_n = \sum_{k|n} k(\lambda_k + \mu_k)$, we obtain

$$\prod_{k,l} \exp\left((-\lambda_l - \mu_l)\frac{z_{kl}}{k}\right) = \prod_m \exp\left(-\alpha_m \frac{z_m}{m}\right)$$

and

$$\prod_{k,l} \exp\left(g_{l,t}\frac{z_{kl}}{k}\right) = \prod_m \exp\left(t^m \frac{z_m}{m}\right).$$

We use these equalities to reorganize the terms in the integral and use the development of the exponential to integrate with respect to the variables z_m.

(3) We evaluate the preceding result in $t = 0$. Therefore, let us remark that when n is chosen such that $n(\lambda_n + \mu_n) = \alpha_n$, the term associated with n in the product (21) is :

$$(-1)^{\lambda_n + \mu_n - 1} n^{\mu_n} \left(\alpha_n - g_{n,t}\right)^{\lambda_n - 1} g_{n,t}(\lambda_n(1 - n) + \alpha_n) \prod_{k=1}^{\mu_n - 1} (\lambda_n + k - g_{n,t})$$

This term is divisible by t and the limit of $\frac{g_{n,t}}{t}$ when t tends to 0 is $\frac{\mu(n)}{n}$. We thus obtain the result.

$\square$

Remark 3.2. When $\lambda_m = 0$ for all m, we obtain the character for the action of the symmetric group on the homology of partition posets, computed by R. Stanley [Sta82] and P. Hanlon [Han81], which was linked with the operad Lie by [Joy86] and [Fre04] (see also [Fre04] for more bibliographical details).

When $\mu_m = 0$ for all m, we obtain the result of F. Chapoton and B. Vallette [CV06] on the homology of pointed partition posets : the character for the action of the symmetric group is linked with the operad PreLie.

4 INCIDENCE HOPF ALGEBRA

We apply in this section the construction detailed in [Sch94] of W. Schmitt of an incidence Hopf algebra associated to a family of posets satisfying some axioms. We then compute the coproduct in this Hopf algebra to identify the studied Hopf algebra with an Hopf algebra of generating series, which will enable us to compute some characteristic polynomials.

4.1 Generalities on incidence Hopf algebra All the definitions recalled here are extracted from the article of W. Schmitt [Sch94].

A family of intervals $\mathcal{P}$ is *interval closed*, if it is non-empty and, for all $P \in \mathcal{P}$ and $x \leq y \in P$, the interval $[x, y]$ belongs to $\mathcal{P}$. An *order compatible relation* on an interval closed family $\mathcal{P}$ is an equivalence relation $\sim$ such

that $P \sim Q$ if and only if there exists a bijection $\phi : P \to Q$ such that $[0_P, x] \sim [0_Q, \phi(x)]$ and $[x, 1_P] \sim [\phi(x), 1_Q]$, for all $x \in P$. Isomorphism of posets is an example of order compatible relation.

Given $\sim$ an order compatible relation on an interval closed family $\mathcal{P}$, we consider the quotient set $\mathcal{P}/\sim$ and denote by $[P]$ the $\sim$-equivalence class of a poset $P \in \mathcal{P}$. We define a $\mathbb{Q}$-coalgebra $C(\mathcal{P})$ as follow:

Proposition 4.1. [Sch94, Theorem 3.1] *Let $C(\mathcal{P})$ denote the free $\mathbb{Q}$-module generated by $\mathcal{P}/\sim$. We define linear maps $\Delta : C(\mathcal{P}) \to C(\mathcal{P}) \otimes C(\mathcal{P})$ and $\epsilon : C(\mathcal{P}) \to \mathbb{Q}$ by:*

$$\Delta[P] = \sum_{x \in P} [0_P, x] \otimes [x, 1_P]$$

and

$$\epsilon[P] = \delta_{|P|,1},$$

where $\delta_{i,j}$ is the Kronecker symbol. Then, $C(\mathcal{P})$ is a coalgebra with comultiplication Δ and counit ϵ.

The *direct product* of posets P_1 and P_2 is the cartesian product $P_1 \times P_2$ partially ordered by the relation $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_i \leq y_i$ in P_i, for $i = 1, 2$. A *hereditary family* is an interval closed family which is also closed under formation of direct products. Let $\sim$ be an order compatible relation on $\mathcal{P}$ which is also a semigroup congruence, i.e., whenever $P \sim Q$ in $\mathcal{P}$, then $P \times R \sim Q \times R$ and $R \times P \sim R \times Q$, for all $R \in \mathcal{P}$. This relation is *reduced* if whenever $|R| = 1$, then $P \times R \sim R \times P \sim P$: all trivial intervals are then equivalent and yields to a unit element for the product on the quotient. These hypotheses ensure that the product will be well defined on the quotient. The obtained unit is denoted by ν. An order compatible relation on a hereditary family $\mathcal{P}$ which is also a reduced congruence is called a *Hopf relation* on $\mathcal{P}$. Isomorphism of posets is a Hopf relation for instance.

Proposition 4.2 ([Sch87]). *Let $\sim$ be a Hopf relation on a hereditary family $\mathcal{P}$. Then $H(\mathcal{P}) = (C(\mathcal{P}), \times, \Delta, \nu, \epsilon, S)$ is a Hopf algebra over $\mathbb{Q}$.*

4.2 Description We now consider maximal intervals in the dual $\Pi^*_{n,p}$ of the semi-pointed partition poset $\Pi_{n,p}$. Using the notations introduced in the first section, we will write $\pi^o_{n,p}$ for $\pi^0_{n,p}$ or $\pi^1_{n,p}$. We denote by $\pi_{n,p}$ the least element of $\pi^o_{n,p}$, whose parts are of size 1 and $M^o_{n,p}$ the greatest element in only one part.

The following proposition ensures that the family $\mathcal{F}$ of direct product of maximal intervals in a semi-pointed partition poset is a hereditary family:

Proposition 4.3 (Intervals in semi-pointed partition posets). *Let p be a semi-pointed partition in the poset $\pi^o_{n,p}$. The interval $[p; M^o_{n,p}]$ in $\pi^o_{n,p}$ is isomorphic to a poset of semi-pointed partitions $\pi^o_{j,l}$, where j is the number of parts in p and l is the number of pointed parts in p.*

The interval $[\pi_{n,p}; p]$ is isomorphic to a product of semi-pointed partitions poset with a factor $\pi^1_{n_j,p_j}$ for every pointed part of p of size n_j with p_j

elements pointed in $\pi_{n,p}$ and a factor $\pi^0_{n_j,p_j}$ for every non-pointed part of p of size n_j with p_j elements pointed in $\pi_{n,p}$.

We consider the relation $\equiv$ given by $P \equiv Q$ if P and Q are isomorphic and if the number of pointed and non pointed parts are the same in the greatest and least elements of P and Q. This relation is a Hopf relation. Considering the hereditary family $\mathcal{F}$ and the Hopf relation $\equiv$, we can apply the construction of W. Schmitt presented in [Sch94], and recalled in the above section, to obtain an incidence Hopf algebra $\mathcal{I}$. This family is an algebra over the set of maximal intervals in semi-pointed partition posets.

4.3   Computation of the coproduct Let us describe now more precisely the coproduct in $\mathcal{I}$. Using the decomposition of intervals described in Proposition 4.3, we obtain the following description of the coproduct:

$$\Delta(\pi^o_{n,p}) = \sum_{j=1}^{n} \sum_{\substack{n_1,\ldots,n_j \geq 1, \ p_1,\ldots,p_j \geq 0 \\ \sum_{i=1}^{j} n_i = n \ \sum_{i=1}^{j} p_i = p}} \sum_{\substack{o_1,\ldots,o_j \in \{0;1\} \\ o_i \leq p_i, \\ o \leq \sum_{i=1}^{j} o_i \leq j-1+o}} c^{\mathbf{o}}_{\mathbf{n,p}} \prod_{i=1}^{j} \pi^{o_i}_{n_i,p_i} \otimes \pi^o_{j,\sum_{i=1}^{j} o_i}$$

where $c^{\mathbf{o}}_{\mathbf{n,p}}$ is the number of partitions having j parts, of size $n_1,\ldots,n_j$, with $p_1,\ldots,p_j$ elements in each part pointed in $\pi_{n,p}$ and with the ith part pointed if o_i is 1 and non pointed otherwise.

Counting the numbers of partitions in $c^{\mathbf{o}}_{\mathbf{n,p}}$ gives the following theorem:

Theorem 4.4. *The coproduct in the incidence Hopf algebra $\mathcal{I}$ of semi-pointed partition poset is given by:*

$$(23) \quad \Delta\left(\frac{\pi^o_{k+l,k}}{l!(k-o)!}\right) = \sum_{p+q \geq 1} \sum_{(l_i,k_i)} \prod_{i=1}^{p} \frac{\pi^1_{l_i+k_i,k_i}}{l_i!(k_i-1)!}$$

$$\prod_{i=p+1}^{p+q} \frac{\pi^0_{l_i+k_i,k_i}}{l_i!k_i!} \otimes \frac{\pi^o_{p+q,p}}{q!(p-o)!},$$

where the second sum runs over the $p+q$-tuples $(l_1,\ldots,l_{p+q})$ and $(k_1,\ldots,k_{p+q})$ satisfying $l_1,\ldots,l_p \geq 0$, $l_{p+1},\ldots,l_{p+q} \geq 1$, $\sum_{i=1}^{p+q} l_i = l$, $k_1,\ldots,k_p \geq 1$, $k_{p+1},\ldots,k_{p+q} \geq 0$ and $\sum_{i=1}^{p+q} k_i = k$.

Proof. In the non pointed case ($o = 0$), the coefficient $c^{\mathbf{o}}_{\mathbf{n,p}}$ is given by:

$$\frac{k!l!\prod_{i=1}^{p} k_i}{\prod_{i=1}^{p+q} k_i!l_i!p!q!}.$$

Indeed, we make $p + q$ packets of elements and the first p packets have to be pointed. For the pointed case ($o = 1$), to ensure that the greatest part is pointed, for instance in 1, we fix that the first packet is pointed in 1. The coefficient $c^{\mathbf{o}}_{\mathbf{n,p}}$ is then given by:

$$\frac{(k-1)!l!\prod_{i=2}^{p} k_i}{(k_1-1)!\prod_{i=2}^{p+q} k_i!l_i!p!q!}.$$

$\square$

4.4   Computation of characteristic polynomials of maximal intervals in the poset We now give an interpretation of the computation of the coproduct in the previous subsection. This interpretation will help us computing characteristic polynomials.

Proposition 4.5. *The incidence Hopf algebra $\mathcal{I}$ of the family of semi-pointed partition posets is isomorphic to the Hopf algebra structure on the polynomial algebra in the variables $(a_{k,l}^o)_{k,l\geq 1,o\in\{0,1\}}$ given by the composition of pairs (F,G) of formal series of the following form:*

$$F = x + \sum_{l,k\geq 1} a_{k,l}^0 \frac{x^k}{k!}\frac{y^l}{l!}$$

$$G = y + \sum_{l,k\geq 1} ka_{k,l}^1 \frac{x^k}{k!}\frac{y^l}{l!}.$$

As a corollary, the Möbius numbers of the intervals $\pi_{n,p}^0$ and $\pi_{n,p}^1$ are respectively the coefficients of A and B, where (A,B) satisfy:

$$(e^B - 1)e^A = x$$

$$Ae^{A+B} = y.$$

By comparison with equations of Proposition 3.3, we obtain an other proof that $A = \mathbf{C}_{-1}^\bullet$ and $B = \mathbf{C}_{-1}^\times$.

We recall the definition of the characteristic polynomial of an interval P whose minimum is denoted by $\hat{0}$ and whose maximum is denoted by $\hat{1}$:

$$\sum_{x\in P} \mu(\hat{0},x)t^{rk(\hat{1})-rk(x)},$$

where $rk(x)$ is the rank of x, i.e. the length of the longest chain between $\hat{0}$ and x.

The characteristic polynomials $\chi^\bullet$ and $\chi^\times$ of $\pi_{n,p}^1$ and $\pi_{n,p}^0$ are then given by:

$$\chi^\bullet = \frac{(e^{t\mathbf{C}_{-1}^\times} - 1)e^{t\mathbf{C}_{-1}^\bullet}}{t}$$

$$\chi^\times = \mathbf{C}_{-1}^\bullet e^{t(\mathbf{C}_{-1}^\bullet + \mathbf{C}_{-1}^\times)}$$

The above relations implies the following proposition:

Proposition 4.6. *The characteristic polynomial of the interval $\pi_{n,p}^1$ is given by:*

$$(24) \qquad\qquad (t-1)\,(t-p-\ell)^{p-2} \prod_{i=p+1}^{p+\ell} (t-i)\,.$$

The characteristic polynomial of the poset $\Pi_{n,p}$ is given by:

$$(25) \qquad (t - p - \ell)^{p-1} \prod_{i=p+1}^{p+\ell} (t - i)\,.$$

The characteristic polynomial of the poset $\widehat{\Pi_{n,p}}$ is given by:

$$(26) \qquad t \times (t - p - \ell)^{p-1} \prod_{i=p+1}^{p+\ell} (t - i) - (1 - p - \ell)^{p-1} \prod_{i=p+1}^{p+\ell} (1 - i)\,.$$

Remark 4.1. The obtained characteristic polynomial is linked with the cycle index series 21 computed previously.

Proof. • We use the following relation between $\mathbf{C}^{\bullet}_{-1}$ and $\mathbf{C}^{\times}_{-1}$, obtained from Proposition 3.3:

$$(27) \qquad e^{\mathbf{C}^{\bullet}_{-1} + \mathbf{C}^{\times}_{-1}} = y + e^{\mathbf{C}^{\bullet}_{-1}}\,.$$

Then, the coefficient of $x^l y^p$ in $\chi^{\bullet}$ is given by the following residue:

$$I = \iint \mathbf{C}^{\bullet}_{-1}(y + e^{\mathbf{C}^{\bullet}_{-1}})^t \frac{dx}{x^{p+1}} \frac{dy}{y^{\ell+1}}\,.$$

We use the substitution $z = \mathbf{C}^{\bullet}_{-1}$ to obtain:

$$I = \iint \frac{z(y + e^z)^t}{(y + e^z)^{p+1}} \left((y + e^z) + z e^z\right) \frac{dz}{z^{p+1}} \frac{dy}{y^{\ell+1}}$$

$$I = \iint z(y + e^z)^{t-p} \frac{dz}{z^{\ell+1}} \frac{dy}{y^{\ell+1}} + \iint z^2 (y + e^z)^{t-p-1} e^z \frac{dz}{z^{p+1}} \frac{dy}{y^{\ell+1}}\,.$$

We expand $(y + e^z)^{t-p}$ and $(y + e^z)^{t-p-1}$ and take the coefficient of y^ℓ. The integral I is then given by:

$$I = \int \binom{t - p}{\ell} e^{(t-p-\ell)z} \frac{dz}{z^p} + \int \binom{t - p - 1}{\ell} e^{(t-\ell-p)z} \frac{dz}{z^{p-1}}\,.$$

As $\int e^{az} \frac{dz}{z^n} = \frac{a^{n-1}}{(n-1)!}$, this gives the result.

• The generating series $\mathcal{S}$ of such characteristic polynomials can be seen as a generating series in t whose coefficients are generating series in x and y. Then, the coefficient of $t^{\ell-1}$ is the sum of Möbius numbers of partitions of $\Pi_{n,p}$ in p parts, weighted by $\frac{x^p y^\ell}{p!\ell!}$. We thus obtain the relation:

$$\mathcal{S} = \frac{e^{t\left(\mathbf{C}^{\bullet}_{-1} + \mathbf{C}^{\times}_{-1}\right)} - 1}{t}\,.$$

The result is obtained by applying Lagrange inversion formula to this relation.

• The characteristic polynomial of $\widehat{\Pi_{n,p}}$ is divisible by $(t - 1)$, as the poset is bounded, and only differs from $t\mathcal{S}$ by its constant term.

$\square$

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